

NORMAL CONDITIONAL EXPECTATIONS OF FINITE INDEX AND SETS OF MODULAR GENERATORS

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ABSTRACT. Normal conditional expectations $E : M \rightarrow N \subseteq M$ of finite index on von Neumann algebras M with discrete center are investigated to find an estimate for the minimal number of generators of M as a Hilbert N -module. Analyzing the case of M being finite type I with discrete center we obtain that these von Neumann algebras M are always finitely generated projective N -modules with a minimal generator set consisting of at most $[K(E)]^2$ modular generators, where $[\cdot]$ denotes the integer part of a real number and $K(E) = \inf\{K : K \cdot E - \text{id}_M \geq 0\}$. This result contrasts with remarkable examples by P. Jolissaint and S. Popa showing the existence of normal conditional expectations of finite index on certain type II_1 von Neumann algebras with center l_∞ which are not algebraically of finite index, cf. Y. Watatani. We show that estimates of the minimal number of modular generators by a function of $[K(E)]$ cannot exist for certain type II_1 von Neumann algebras with non-trivial center.

1. Introduction. A conditional expectation $E : M \rightarrow N \subseteq M$ is said to be of finite index if there exists a finite constant $K(E) = \inf\{K : K \cdot E - \text{id}_M \geq 0\}$, cf. [15, 8]. The goal of the present paper is the investigation of minimal generating sets of right Hilbert W^* -modules arising as $\{M, E(\langle \cdot, \cdot \rangle_M)\}$ for conditional expectations $E : M \rightarrow N \subseteq M$ of finite index on von Neumann algebras M with discrete center $Z(M)$ (where $\langle x, y \rangle_M = x^*y$ for every $x, y \in M$). We describe this case and indicate partial solutions for the complementary case of von Neumann algebras with diffuse center. The case of finite von Neumann algebras M with discrete center proves to be of major importance. Most interesting examples of normal conditional expectations on von Neumann algebras of type II_1 with center l_∞ that are non-algebraically of finite index were discovered by P. Jolissaint [9, Prop. 3.8] and by S. Popa [16, Rem. 2.4]. We give a detailed description of the situation and show the strong bounds of this phenomenon. In particular, type II_1 von Neumann algebras M with non-trivial finite-dimensional center are always finitely generated projective N -modules, however the minimal number of modular generators cannot be estimated by a function of the characteristic constant $K(E)$, in general.

Analyzing the case of von Neumann algebras M which are direct integrals of a field of finite-dimensional type I factors over their discrete center we obtain that these von Neumann algebras are always finitely generated projective N -modules with a minimal modular generator set consisting of at most $[K(E)]^2$ generators, where $[\cdot]$ denotes the integer part of a real number.

Conditional expectations of finite index on von Neumann algebras have remarkable properties: they are automatically normal and faithful ([16, Prop. 1.1]), the von Neumann algebra M is complete with respect to the Hilbert norm derived from the new N -valued inner product $E(\langle \cdot, \cdot \rangle_M)$ ([2, Prop. 3.3]), and the center of M is finite-dimensional if and only if the center of N is. Moreover, they commute with the abstract projections of von Neumann algebras to most of their type components, especially to the parts with discrete and diffuse center ([3, Th. 1]) and to their type I_{fin} , I_∞ , II_1 , II_∞ and III parts ([15, 1.1.2, (iii)], [8, §2]). This justifies the canonical character of our basic question for minimal generating sets of M as an N -module and opens up the chance to solve it in the form of separate case studies of the different types of von Neumann algebras M that can occur.

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The case of properly infinite von Neumann algebras M was solved by several authors without any reference to special structures of the center: there exists a single element $m \in M$ such that every element $x \in M$ can be represented as $x = mE(m^*x)$ and $E(m^*m) = 1_M$, cf. [2, Lemma 3.21, Prop. 3.22], [15, Th. 1.1.6, (b)]. The proofs show the existence of $m \in M$. We provide a constructive proof of the generating element m for properly infinite discrete type I von Neumann algebras during the proof of Theorem 4 below, cf. the final remarks.

2. The type II_1 case. Due to the type splitting properties of E we can concentrate our efforts on the finite case in the sequel. If M and N are type II_1 factors then M is a finitely generated projective N -module with a minimal set of modular generators consisting of at most $[K(E)] + 1$ elements, cf. [13, Prop. 1.3]. For type II_1 von Neumann algebras with discrete center finite generation of M as an N -module can only be guaranteed over finite-dimensional pieces of the center of N , cf. [1].

Proposition 1. *If $E : M \rightarrow N \subseteq M$ is a conditional expectation of finite index on a type II_1 von Neumann algebra M with discrete center then for any finite number of minimal central projections of N with sum $p \in Z(N)$ the right Hilbert pN -module Mp is finitely generated.*

Proof. By [2, 8] the von Neumann algebra $N = E(M)$ and the set $\text{End}_N(M)$ of all bounded N -linear operators on the Hilbert N -module $\{M, E(\langle \cdot, \cdot \rangle_M)\}$, which is a von Neumann algebra, have to be of type II_1 with discrete center. Both these W^* -algebras share a common center $Z(N)$ if we identify appropriate multiples of the identity operator on M with the corresponding elements of $Z(N)$. Selecting a finite sum p of minimal central projections of $Z(N)$ the corresponding von Neumann algebra $p \cdot \text{End}_N(M)$ does not possess any non-unital two-sided norm-closed ideal. In particular,

$$p \cdot \text{End}_N(M) \equiv \text{lin}\{p \cdot \theta_{x,y} : \theta_{x,y}(z) = yE(x^*z) \text{ for } z \in M\}.$$

By [19, Ex. 15.O] the Hilbert pN -module $\{M, E(\langle \cdot, \cdot \rangle_M)\}$ has to be finitely generated. \square

Surprisingly, the statement on type II_1 von Neumann algebras M to be finitely generated projective N -modules for conditional expectations $E : M \rightarrow N \subseteq M$ of finite index, in general fails to be true, (cf. [3, Prop. 1]). A remarkable counterexample to this conjecture was brought to my attention by S. Popa, cf. [16, Rem. 2.4]. A similar construction with a different proof can be found in a paper by P. Jolissaint [9, Th. 2.2, Prop. 3.8]. The counterexample also shows in its generalized version that an analogue of Proposition 1 cannot be true for type II_1 von Neumann algebras with diffuse center even if it is generated by a non-discrete finite measure space.

Example 2. Let R be the hyperfinite type II_1 W^* -factor and select a projection p_k with $\text{tr}(p_k) = 2^{-k}$. Since the type II_1 factors $p_k R p_k$ and $(1 - p_k)R(1 - p_k)$ are $*$ -isomorphic we can find a $*$ -isomorphism θ_k of them. Set $M_k = \{x + y : x \in p_k R p_k, y \in (1 - p_k)R(1 - p_k)\}$ and $N_k = \{x + \theta_k(x) : x \in p_k R p_k\}$. Note that $p_k M_k p_k = N_k p_k$, $(1 - p_k)M_k(1 - p_k) = N_k(1 - p_k)$ and $N'_k \cap M_k = \mathbb{C}p_k \oplus \mathbb{C}(1 - p_k)$.

Considering the unique trace-preserving conditional expectation $E_{k,tr} : M_k \rightarrow N_k \subset M_k$ with $k \in \mathbb{N}$ we obtain $\text{Ind}(E_{k,tr}) = \text{tr}(p_k)^{-1} + \text{tr}(1 - p_k)^{-1} = 2^k + 2^k/(1 - 2^k)$ by [11, §2.5, Th., (iii)], [10, Lemma 2.2.2]. Following [2, Prop. 3.15] any other conditional expectation $E_k : M_k \rightarrow N_k \subset M_k$ can be expressed as $E_k(x) = E_{k,tr}(a^*xa)$ for some element $a \in N'_k \cap M_k$. Among these conditional expectations we can find one of minimal index, $E_{k,min}$, with index value $\text{Ind}(E_{k,min}) = 4$ independent of k . Two specific values are $E_{k,min}(p_k) = E_{k,min}(1 - p_k) = 1/2$. Obviously, M_k is a finitely generated projective N_k -module for every $k \in \mathbb{N}$. However, the number of modular generators has to be greater than 2^k since every generator $u_k \in M_k$ can contribute at most the value 1 to the index value $\text{Ind}(E_{k,tr}) = 2^k + 2^k/(1 - 2^k)$ of the trace-preserving conditional expectation $E_{k,tr}$ because of the finiteness of M_k , cf. [2, Th. 3.5]. Indeed,

$\text{tr}(u_k u_k^*) = \text{tr}(u_k^* u_k) = \text{tr}(E_{k, \text{tr}}(u_k^* u_k)) \leq 1$ since the generators u_k can be selected such that $E_{k, \text{tr}}(u_k^* u_k)$ is a projection by [2], and so the spectrum of $u_k u_k^*$ is restricted to a subset of the unit interval $[0, 1]$, i.e. $u_k u_k^* \leq 1$.

In a final step we form W^* -algebras M and N with a common direct integral decomposition over $l_\infty(\mathbf{N})$ attaching to every minimal projection q_k of $l_\infty(\mathbf{N})$ either M_k or N_k , respectively. The unique trace-preserving conditional expectation $E_{\text{tr}} : M \rightarrow N \subset M$ is not of finite index any longer, whereas the minimal conditional expectation $E_{\text{min}} : M \rightarrow N \subset M$ has the index value $\text{Ind}(E_{\text{min}}) = 4$. Since the number of generators of M_k as a finitely generated projective N_k -module is greater than 2^k the total number of generators of M as a self-dual Hilbert N -module is infinite.

We can find analogous examples for type II_1 von Neumann algebras M with diffuse center. To see this take the diffuse von Neumann algebra $L^\infty([0, 1], \lambda)$ and form two direct integrals over $[0, 1]$ in such a way that all fibres on the intervals $[1/(k+1), 1/k]$ are $*$ -isomorphic to M_k or N_k , respectively, ($k = 1, 2, \dots$). Denote the resulting von Neumann algebras by M and N , respectively. The von Neumann algebra M cannot be a finitely generated Hilbert N -module for the same reasons as for the example with discrete center.

The index value 4 is not essential and can be replaced by any other number within the range $[4, \infty)$ since we can select another conditional expectation E with a fixed finite index between $\text{Ind}(E_{\text{tr}}) = (2^{2k} - 2^{k+1})/2^k - 1 \xrightarrow{k \rightarrow \infty} \infty$ and $\text{Ind}(E_{\text{min}}) = 4$ relying on suitable elements in the non-trivial relative commutants. The number of modular generators is not affected by this choice. For index values lower than four the relative commutant has to be trivial, and we are again in the factor case.

Corollary 3. *Let $E : M \rightarrow N \subseteq M$ be a (normal) conditional expectation of finite index on a W^* -algebra M with image algebra N . Then M is not necessarily a finitely generated projective N -module, especially if N is of type II_1 with infinite-dimensional discrete or arbitrary diffuse center.*

Furthermore, even if M has a finite-dimensional non-trivial center the minimal number of generators of M as a finitely generated projective N -module cannot be estimated by a fixed function of $K(E)$, in general.

3. The discrete finite case. The remaining case is that of a von Neumann algebra M of type I with discrete center, but without properly infinite part in its central direct integral decomposition. The situation for finite-dimensional centers was partially considered by T. Teruya [17]. For our purposes the center of M can be arbitrarily large. Also, the relative position of the centers $Z(M)$ and $Z(N)$ can be arbitrary, i.e. somewhere between the simplest case $Z(M) = Z(N)$ and the most complicated case $Z(M) \cap Z(N) = \mathbf{C}1_M$, see [8] for examples.

Theorem 4. *Let M be a discrete W^* -algebra without properly infinite part. Consider a (normal) conditional expectation $E : M \rightarrow N \subseteq M$ of finite index possessing the structural constant $1 \leq K(E) = \inf\{K : K \cdot E - \text{id}_M \geq 0\} \in \mathbf{R}$. Then M is a finitely generated projective N -module with a minimal modular generator set consisting of at most $[K(E)]^2$ generators, where $[\cdot]$ denotes the integer part of a real number.*

If $n = K(E)$ is an integer then the normalized trace E on the full $n \times n$ -matrix algebra $M = M_n(\mathbf{C})$ realizes the indicated upper bound n^2 for the number of modular generators, i.e. the estimate can in general not be improved.

Proof. We note in passing that the construction principles do not depend on whether we consider M as a right or left N -module. To be precise we use the right N -module notation subsequently. Moreover, the considerations below work equally well for the properly infinite discrete case for which a stronger result is already known. So they are only of interest as a way to construct the generating element $m \in M$ for this case.

The strategy of the presented proof is to find a suitable set of modular generators in the beginning, and to reduce this set of generators until we obtain a sufficiently small generating set fulfilling the prediction. Note that one or the other reduction step might be redundant when we resort to particular examples, however the stressed for statement requires a proof from a general viewpoint.

Consider a minimal (with respect to N) projection $p \in N$. Let $q \leq p$ be a projection of M . Since E is faithful $E(q) \neq 0$ and the inequality $0 < E(q) \leq E(p) = p$ holds. Because of the minimality of p inside N there exists a number $\mu \in (0, 1]$ such that $E(q) = \mu p$. That is,

$$E(\mu^{-1} \cdot q) = p.$$

Moreover, since E was supposed to be of finite index with structural constant $K(E) \geq 1$ we obtain

$$K(E) \cdot p = K(E) \cdot E(\mu^{-1} q) \geq \mu^{-1} q > 0$$

and hence, the estimate $\mu \geq K(E)^{-1}$. Suppose, $p \in N$ can be decomposed into a sum of pairwise orthogonal (arbitrary) projections $\{q_\alpha : \alpha \in I\}$ inside M . Obviously, $q_\alpha \leq p$ for every $\alpha \in I$, and

$$p = E(p) = \sum_{\alpha \in I} E(q_\alpha) = \left(\sum_{\alpha \in I} \mu_\alpha \right) p \geq \left(\sum_{\alpha \in I} K(E)_{(\alpha)}^{-1} \right) p.$$

Consequently, the sum has to be finite and the maximal number of non-trivial summands is $[K(E)]$, the integer part of $K(E)$. We see, for the minimal projections $p \in N \subseteq M$ every family of pairwise orthogonal subprojections $\{q_i\} \in M$ of p is finite.

In this way we obtain that every minimal projection $p \in N$ has to be represented only in a small part of the central direct integral decomposition of M which has a finite-dimensional center. Extending a minimal projection $p \in N \subseteq M$ by the partial isometries of $N \subseteq M$ every W^* -factor block of the central direct integral decomposition of N turns out to be represented in a part of the central direct integral decomposition of M with a finite-dimensional center. Conversely, the conditional expectation of finite index E maps each W^* -factor block of the central direct integral decomposition of M into a part of the central direct integral decomposition of N with a finite-dimensional center. To see that consider a minimal central projection $q \in M$ together with the conditional expectation $E' : qM \rightarrow qN \subseteq qM$, where $E'(x) = E(x) \cdot E(q)^{-1} \cdot q$ for $x \in qM$. Since $K(E) \cdot E(q) \geq q$ by supposition we obtain

$$K(E) \cdot E(x) \cdot E(q)^{-1} \cdot q \geq K(E) \cdot E(x) \cdot q \geq q x q = x q$$

for any $x \in qM$. Therefore, $K(E') \leq K(E)$, and $\dim(Z(qN)) \leq [K(E)]$ by [2, Th. 3.5, (ii)] and by the structure of a qN -module basis of the Hilbert qN -module $\{qM, E'(\langle \cdot, \cdot \rangle)\}$.

To find a suitable set of generators of M as a right Hilbert N -module we decompose the identity $1_N = 1_M$ into a w^* -sum of pairwise orthogonal minimal projections $\{p_\nu\} \subset N$, and further into a suitable subdecomposition of this sum into a w^* -sum of pairwise orthogonal minimal projections $\{q_\alpha : \alpha \in I\} \subset M$. Without loss of generality we will always assume that minimal projections $p_{\nu_1}, p_{\nu_2} \in N$ of our choice with $p_{\nu_1} = v^* v$, $p_{\nu_2} = v v^*$ for $v \in N$ possess finite sum decompositions $p_{\nu_1} = \sum_i q_{i, \nu_1}$, $p_{\nu_2} = \sum_i q_{i, \nu_2}$ with finite sets of minimal projections $\{q_{i, \nu_1}\}, \{q_{i, \nu_2}\} \subset M$ such that every partial isometry $q_{i, \nu_1} v \in M$ has the corresponding domain projection q_{i, ν_2} .

In our special setting a suitable N -module basis of M contains this maximal set of pairwise orthogonal minimal projections $\{q_\alpha\}$ of M scaled down by the inverse of the number μ_α arising from the equality $E(q_\alpha) = \mu_\alpha p_\nu$ for some minimal projection $p_\nu \in N$ of our initial choice and certain $\mu_\alpha \in [K(E)^{-1}, 1]$. As an intermediate result we get

$$\{\mu_\alpha^{-1/2} \cdot q_\alpha : \alpha \in I\} \subseteq \text{basis}.$$

If M is commutative, then the Hilbert N -module basis of M is complete, and

$$\text{Ind}(E) = \sum_{\alpha \in I} \mu_{\alpha}^{-1/2} \cdot q_{\alpha} \cdot (\mu_{\alpha}^{-1/2} \cdot q_{\alpha})^* \leq K(E) \sum_{\alpha \in I} q_{\alpha} \leq K(E) \cdot 1_M$$

by [2, Thm. 3.5]. However, if M is non-commutative, then we have to add all those minimal partial isometries $\{u_{\beta} : \beta \in J\}$ of M that connect two minimal projections of our choice $\{q_{\alpha}\}$, but also scaled down by the inverse of the number μ_{β} arising from the equality $E(u_{\beta}^* u_{\beta}) = \mu_{\beta} p_{\nu}$ for some minimal projection $p_{\nu} \in N$ of our initial choice and certain $\mu_{\beta} \in [K(E)^{-1}, 1]$, cf. [8, Ex. 1.1]. Finally,

$$(1) \quad \{\mu_{\alpha}^{-1/2} \cdot q_{\alpha} : \alpha \in I\} \cup \{\mu_{\beta}^{-1/2} \cdot u_{\beta} : \beta \in J\} \equiv \text{basis}.$$

Finally a Hilbert N -module basis of M is complete, but often rather large.

In a next step we use Hilbert N -module isomorphisms to reduce the number of generators in the generating set (1). For simplicity we consider the projections $\{q_{\alpha}\}$ as special partial isometries, too. Define equivalence classes of partial isometries of (1) by the rule: $u_{\beta} \sim u_{\gamma}$ if and only if $q_{\alpha} u_{\beta} \neq 0$, $q_{\alpha} u_{\gamma} \neq 0$ for a certain minimal projection $q_{\alpha} \in M$ of our choice (1) and $u_{\beta} = u_{\gamma} v$ for some partial isometry $v \in N$ which links two projections $\{p_{\nu}\} \subset N$ of our initial choice. Then the Hilbert N -modules $\{u_{\beta}(\mu_{\beta}^{-1/2} N), E(\langle \cdot, \cdot \rangle_M)\}$ and $\{u_{\gamma}(\mu_{\gamma}^{-1/2} N), E(\langle \cdot, \cdot \rangle_M)\}$ derived from equivalent partial isometries must be unitarily isomorphic, since the set identity $u_{\gamma} N \equiv u_{\gamma} v(v^* N) \equiv u_{\beta} N$ holds inside M .

For every equivalence class we select a representative u_{β} which has to be a minimal projection q_{α} of our choice (1) whenever there is one contained in the equivalence class under consideration. Since N and the direct sum of Hilbert N -modules $w^*\text{-}\sum_{\omega \in I} N_{(\omega)} \subseteq M$ are unitarily isomorphic as Hilbert N -modules (where I has the same cardinality as the index set of the selected sequence of minimal projections $\{q_{\alpha}\} \subset M$), we find that the Hilbert N -submodule $u_{\beta} N$ is unitarily isomorphic to the direct sum of Hilbert N -modules $w^*\text{-}\sum_{\gamma} u_{\gamma} N \subseteq M$, where the index runs over all possible indices γ with the property $u_{\beta} \sim u_{\gamma}$.

Consequently, we can reduce our set of generators (1) in such a way that there only remains one element of every equivalence class of partial isometries, which should be a projection in case the equivalence class contains one, and which should be a partial isometry connecting two of the just selected projections otherwise. For further considerations we will use the same notation as above in (1) to refer to the reduced generator set of M generating it as a right Hilbert N -module.

After this factorization-like procedure we obtain a possibly smaller generating set of M with a very special property: for every minimal projection $q_{\alpha} \in M$ in this generating set there are at most $([K(E)] - 1)$ other minimal projections in it which are connected to q_{α} by a partial isometry of this generating set. Indeed, every minimal projection of this kind has its own minimal N -central carrier projection by construction. Since the W^* -factor $M q_{\alpha} M$ intersects with at most $[K(E)]$ W^* -blocks of the central direct integral decomposition of N as shown above the statement yields. To proceed we fix a minimal projection q_{α} of the actually selected set that generates M as a Hilbert N -module, and we form the sum p_{α} of all those minimal projections of N majorizing a minimal projection of M of our latter choice that is equivalent to q_{α} . This is a finite sum with at most $[K(E)]$ summands, and the W^* -algebra $p_{\alpha} N p_{\alpha}$ is commutative. Therefore, the W^* -algebra $p_{\alpha} M p_{\alpha}$ is a matrix algebra since E restricted to it has the commutative finite-dimensional image $p_{\alpha} N p_{\alpha}$, cf. [8, Cor. 4.4]. Finally, the number of modular generators of our last choice contained in $p_{\alpha} M p_{\alpha}$ cannot exceed $[K(E)]^2$ since the number of minimal projections of M of this choice does not exceed $[K(E)]$ as shown.

By transfinite induction we get a partition of the identity of M (and N) as a sum of pairwise orthogonal projections of type $p_{\alpha} \in N$, and any generator of the made choice has to be contained

in one of the W^* -subalgebras of type $p_\alpha Mp_\alpha$ of M , where the carrier projections of these W^* -subalgebras are pairwise orthogonal in N by construction. Consequently, we can form sum-compositions of generators of our selected minimal set to reduce the number of them further by the following principle: take one generator per W^* -subalgebra of type $p_\alpha Mp_\alpha$ and form the appropriate w^* -sum of them inside M to get a new generator. (Equivalently, we can find a subset of singly generated Hilbert N -submodules of our decomposition of M which can be summed up to another singly generated Hilbert N -submodule reducing the number of direct summands that way enormously.) In fact, we can form at most $[K(E)]^2$ modular generators of this type since every W^* -subalgebra of type $p_\alpha Mp_\alpha$ was shown to contain at most as many of them of our choice made.

Finally, we have constructed a generating set of the Hilbert N -module $\{M, E(\langle \cdot, \cdot \rangle)\}$ which has at most $[K(E)]^2$ modular generators, and the theorem is proved. \square

4. Final remarks. The construction given in the proof works for properly infinite discrete von Neumann algebras equally well. However, there is an isomorphism $H \cong \sum_{i \in I} H_{(i)}$ which exists for every infinite-dimensional Hilbert space H and for $\text{card}(I) \leq \dim(H)$ due to the existence of a chain of pairwise orthogonal projections each of which is similar to 1_M and which sum up to 1_M . So we can again reduce our set of generators to a single generator $m \in M$.

The remaining case to be investigated is the case of finite von Neumann algebras with diffuse center and some obstruction like $Z(M) \cap Z(N) = \mathbf{C}1_M$, at least valid for parts of the centers. This question is still unsolved at present.

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